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What to measure when measuring noise in MRI

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Motivation:

- Many papers and methods to estimate noise out of MRI data.
- Noise estimation vs. SNR estimation
- In single coil systems, variance of noise is a “good” measure.
- Complex systems: what are we really measuring? Is the variance of noise still valid?

Outline

1. Noise in MR acquistions
2. Basic Models
   - Rician
   - Non-central chi
3. More Complex Models
   - Correlated multiple coils
   - Parallel MRI: SENSE and GRAPPA
4. The nc-chi example
   - Non-stationary noise
   - Effective values
5. Other models
10. Conclusions


MR
Positioning Kit

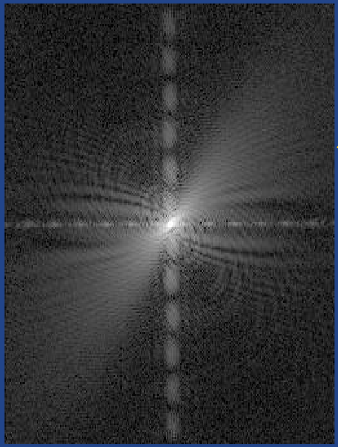
1- Noise in MRI

Signal acquisition (Single coil)

Scanner

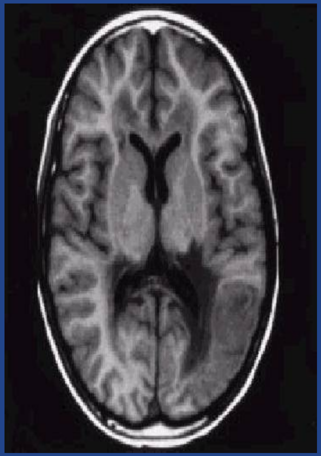
→

K-space



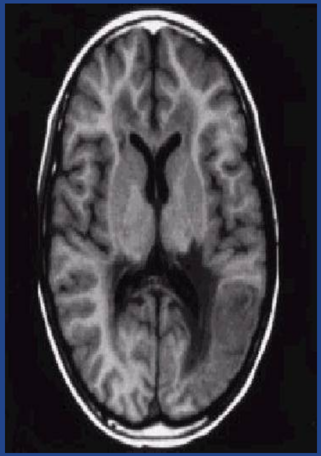
F-1 →

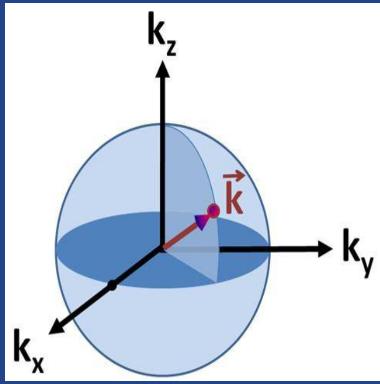
X-space (complex)



| . | →

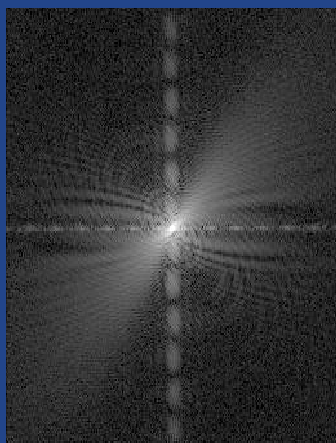
magnitude





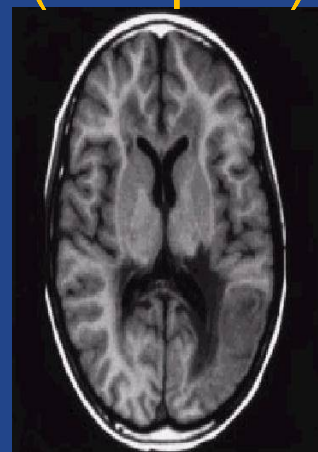
Acquisition Noise (single coil)

K-space



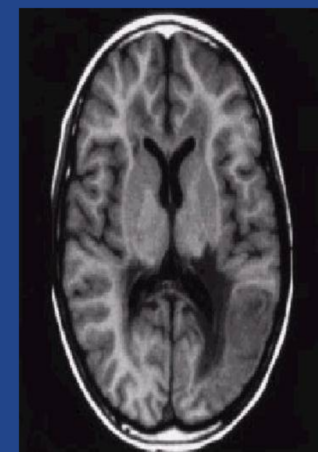
F-1 →

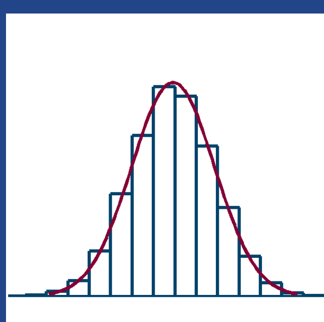
X-space (complex)



| . | →

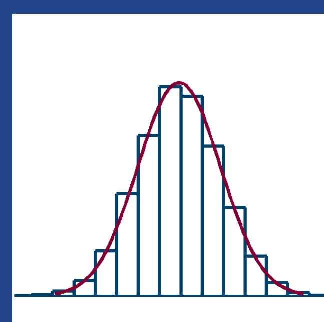
magnitude





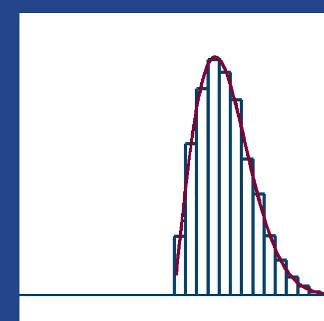
Complex Gaussian

$\sigma2K$



Complex Gaussian

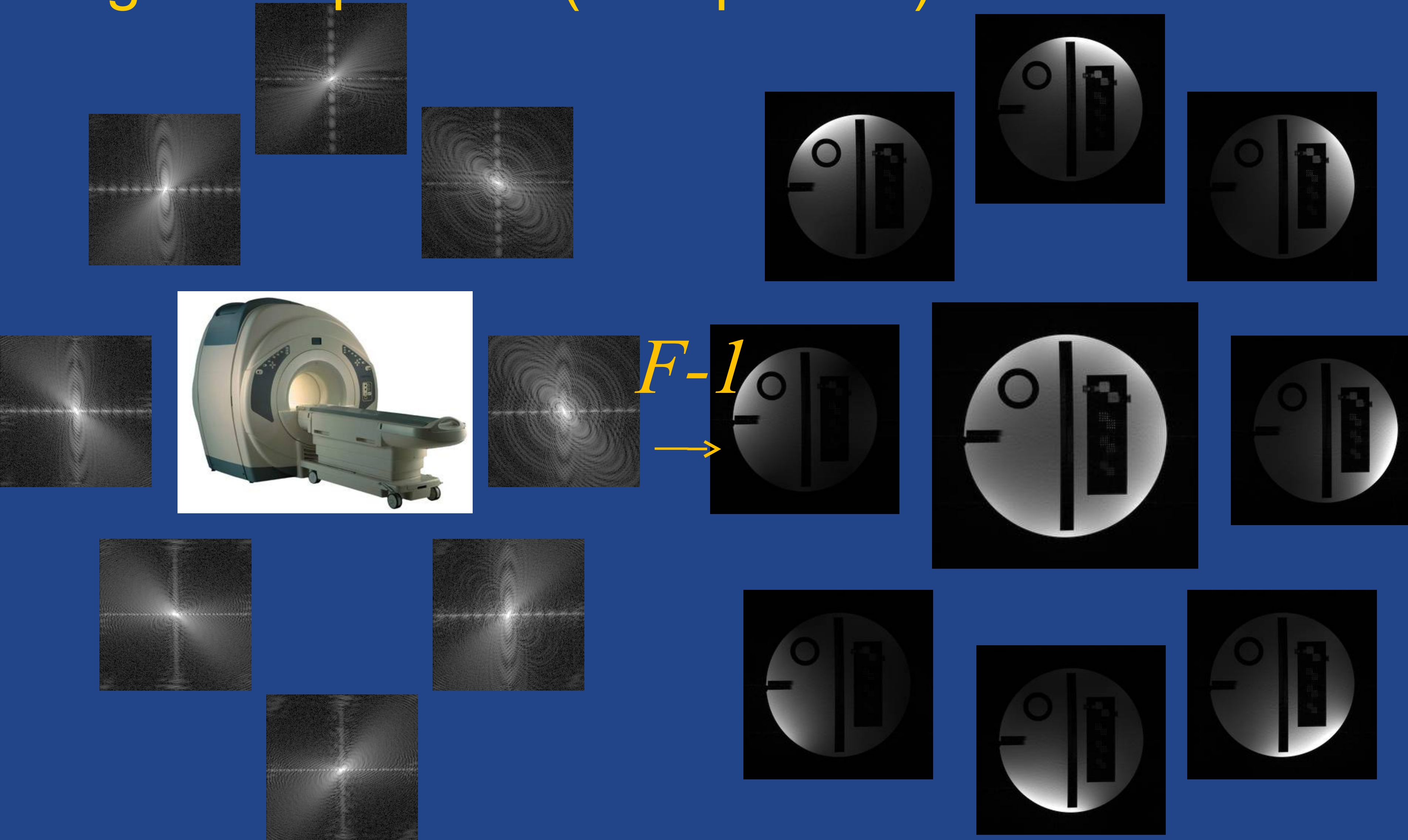
$\sigma2n=\sigma2K /|\Omega|$



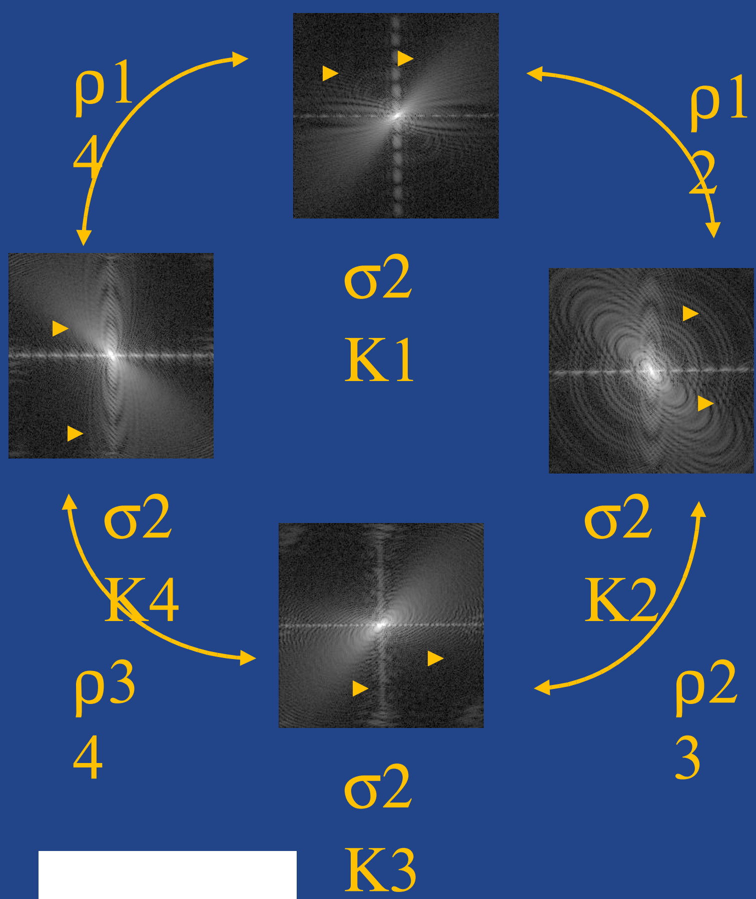
Rician

$\sigma2n$

Signal acquisition (Multiple coil)

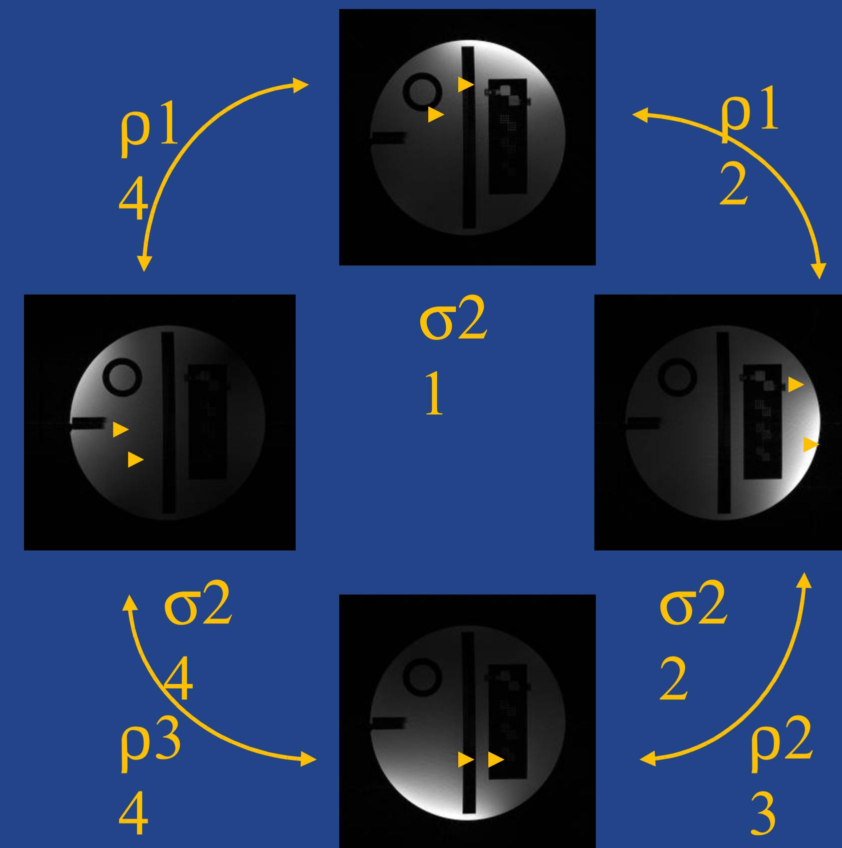


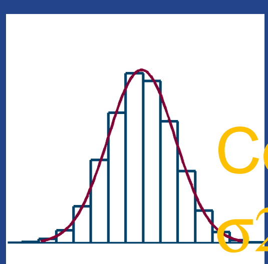
Acquisition noise (Multiple coil)



F-1

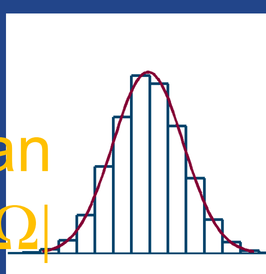
→





Complex Gaussian
σ2Ki

Complex Gaussian
σ2i=σ2Ki /|Ω|



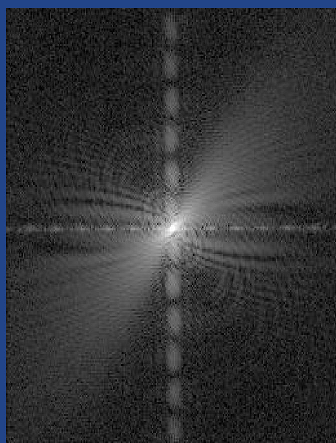
Acquisition noise:

- Noise in receiving coils is complex Gaussian (assuming no post-processing)
- Noise in x-space related to noise in k-space
- Final distribution will depend on the reconstruction

2- Basic Noise Models

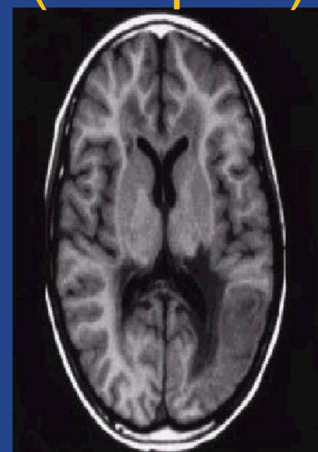
Rician Model

K-space



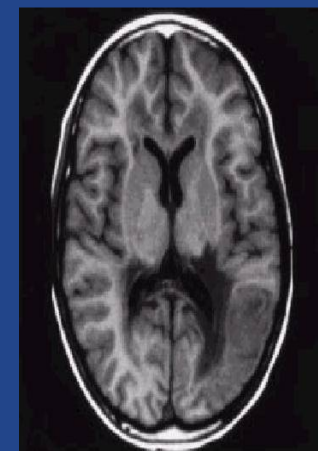
F-1 →

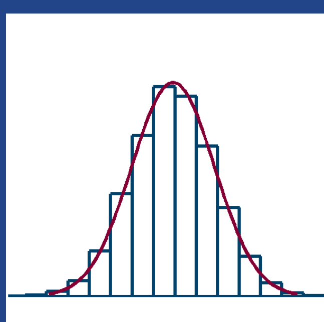
X-space (complex)



| . | →

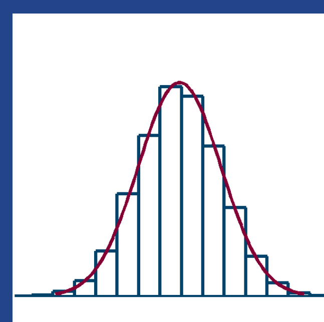
magnitude





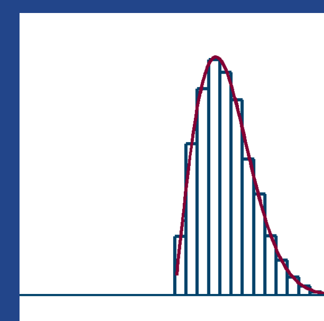
Complex Gaussian

σ^2_K



Complex Gaussian

$\sigma^2_n=\sigma^2_K /|\Omega|$



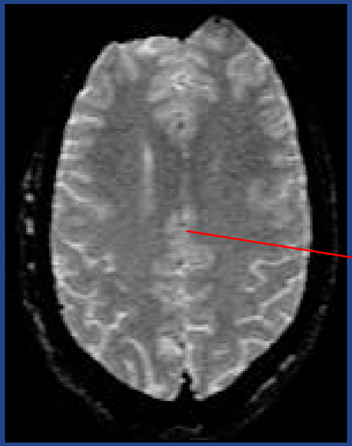
Rician

σ^2_n

Rician Model

Complex $C(\mathbf{x}) = (A_R + n_r(\sigma_n^2)) + j(A_I + n_i(\sigma_n^2))$

Magnitude $M(\mathbf{x}) = \sqrt{(A + n_r(\sigma_n^2))^2 + n_i(\sigma_n^2)^2}$



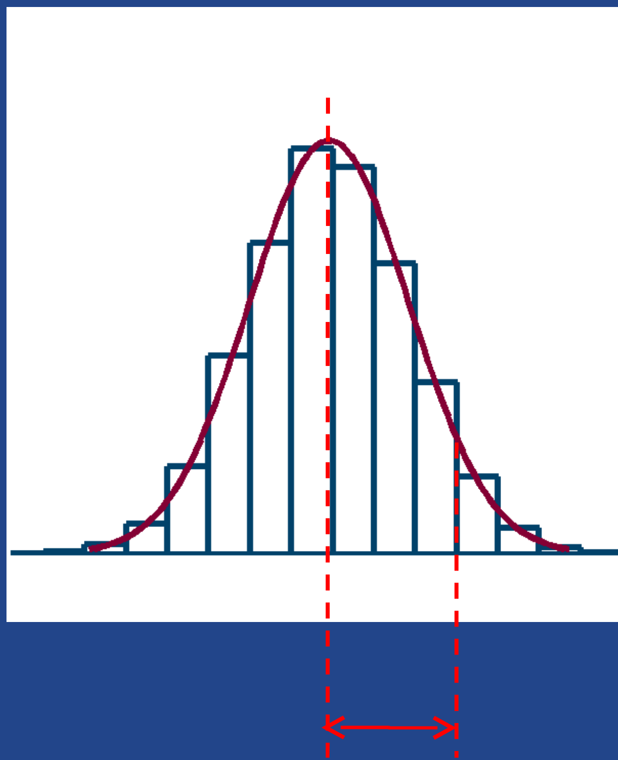
$$p_M(M|A,\sigma_n) = \frac{M}{\sigma_n^2} e^{-\frac{M^2+A^2}{2\sigma_n^2}} I_0\left(\frac{AM}{\sigma_n^2}\right) u(M)$$

Rician

$$p_M(M|\sigma_n) = \frac{M}{\sigma_n^2} e^{-\frac{M^2}{2\sigma_n^2}} u(M)$$

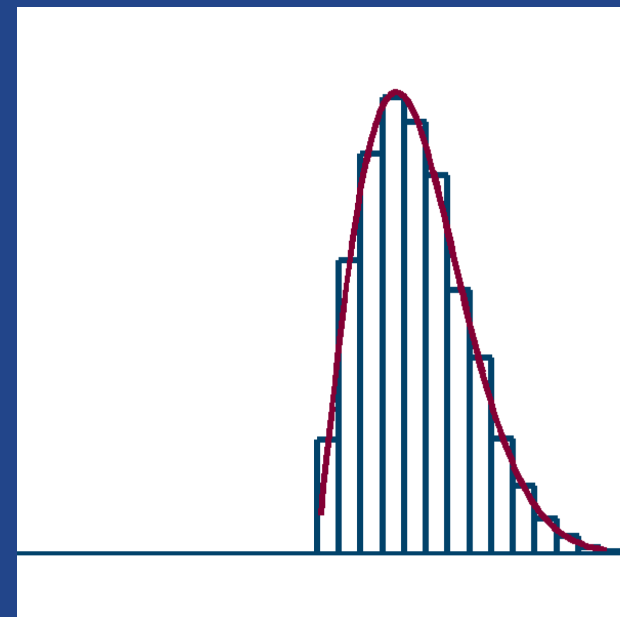
Rayleigh

Gaussian



σn

Rician

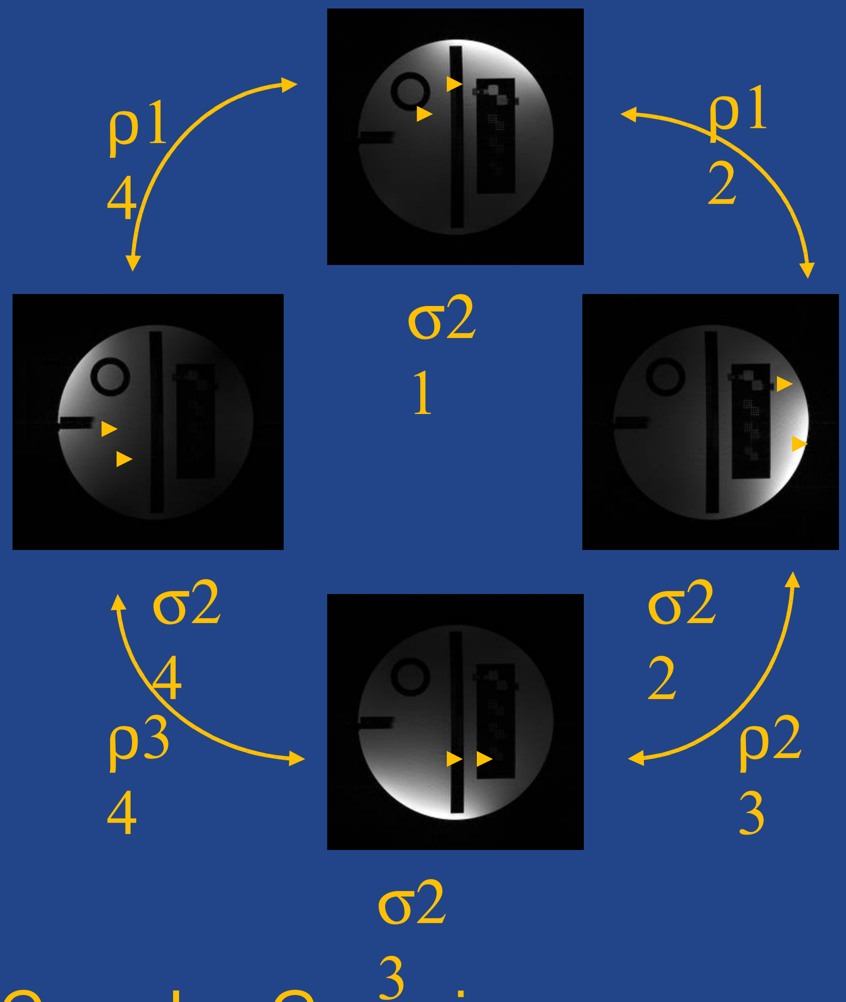


Meaning of σn?

$$p_M(M|A,\sigma_n) = \frac{M}{\sigma_n^2} \, e^{-\frac{M^2+A^2}{2\sigma_n^2}} \, I_0\left(\frac{AM}{\sigma_n^2}\right) \, u(M)$$

SNR=A / σn

Non-central chi model



Particular case:

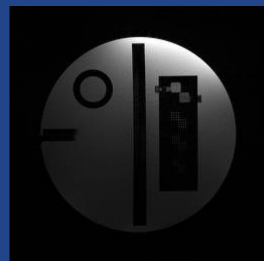
$$\sigma^2_i = \sigma^2_j = \sigma^2_n$$

$$\rho_{ij} = 0$$

Complex Gaussian

$$\sigma^2_i = \sigma^2 K_i / |\Omega|$$

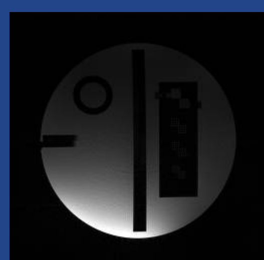
Non-central chi model



σ2
n

σ2
n

σ2
n



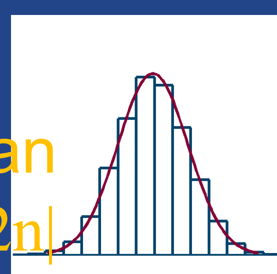
σ2
n

Sum of Squares

$$M_L(\mathbf{x}) = \sqrt{\sum_{l=1}^{L} |S_l(\mathbf{x})|^2}.$$

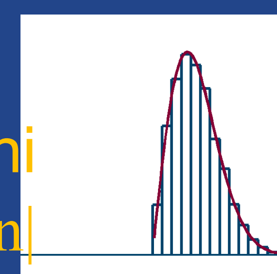
Complex Gaussian

σ2n



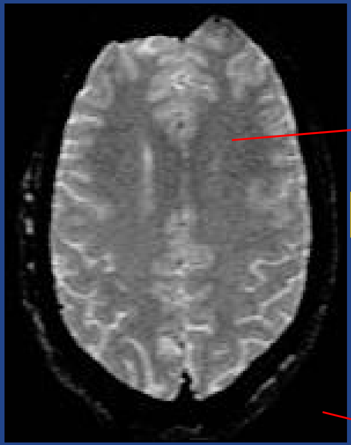
Non central chi

L, σ2n



Non-central chi model

$$M_L(\mathbf{x}) = \sqrt{\sum_{l=1}^{L} |S_l(\mathbf{x})|^2}.$$



$$p_{M_L}(M_L|A_L, \sigma_n, L) = \frac{A_L^{1-L}}{\sigma_n^2} M_L^L e^{-\frac{M_L^2 + A_L^2}{2\sigma_n^2}} I_{L-1}\left(\frac{A_L M_L}{\sigma_n^2}\right) u(M_L)$$

Non Central Chi

$$p_{M_L}(M_L|\sigma_n, L) = \frac{2^{1-L}}{\Gamma(L)} \frac{M_L^{2L-1}}{\sigma_n^{2L}} e^{-\frac{M_L^2}{2\sigma_n^2}} u(M_L)$$

Central Chi

Basic models:

- Rician: relation between $\sigma 2n$ and variance of noise in Gaussian complex data.
- Nc-chi: also relation between $\sigma 2n$ and Gaussian Variance.
- Nc-chi: many times, interesting parameter $\sigma 2n \cdot L$

$$E\{M(x)2\}=A(x)2+2L\cdot\sigma 2n$$

$$SNR=A(x)/L1/2\cdot\sigma n$$

- Usually: equivalence between $L\cdot\sigma 2n$ (nc-chi) and $\sigma 2n$ (Rician).

Example: Conventional approach

$$\hat{A}_c = \sqrt{\max\left(\langle M^2 \rangle - 2\sigma_n^2, 0\right)}$$

Rician

$$\widehat{A_L}(\mathbf{x}) = \sqrt{\max\left(\langle M_L^2(\mathbf{x}) \rangle_{\mathbf{x}} - 2\sigma_{nL}^2, 0\right)}$$

Noncentral-chi

Example: LMMSE filter

$$\widehat{A_{ij}^2} = \langle M_{ij}^2 \rangle - 2\sigma_n^2 + K_{ij} \left(M_{ij}^2 - \langle M_{ij}^2 \rangle \right)$$

$$K_{ij} = 1 - \frac{4\sigma_n^2 \left(\langle M_{ij}^2 \rangle - \sigma_n^2 \right)}{\langle M_{ij}^4 \rangle - \langle M_{ij}^2 \rangle^2}.$$

Rician

$$\widehat{A_L^2(\mathbf{x})} = \langle M_L^2(\mathbf{x}) \rangle_{\mathbf{x}} - 2L\sigma_n^2 + K_L(\mathbf{x}) \left(M_L^2(\mathbf{x}) - \langle M_L^2(\mathbf{x}) \rangle_{\mathbf{x}} \right),$$

$$K_L(\mathbf{x}) = 1 - \frac{4\sigma_n^2 \left(\langle M_L^2(\mathbf{x}) \rangle_{\mathbf{x}} - L\sigma_n^2 \right)}{\langle M_L^4(\mathbf{x}) \rangle_{\mathbf{x}} - \langle M_L^2(\mathbf{x}) \rangle_{\mathbf{x}}^2}.$$

Noncentral-chi

3- More Complex Models

Limitation of the nc-model:

Only valid if:

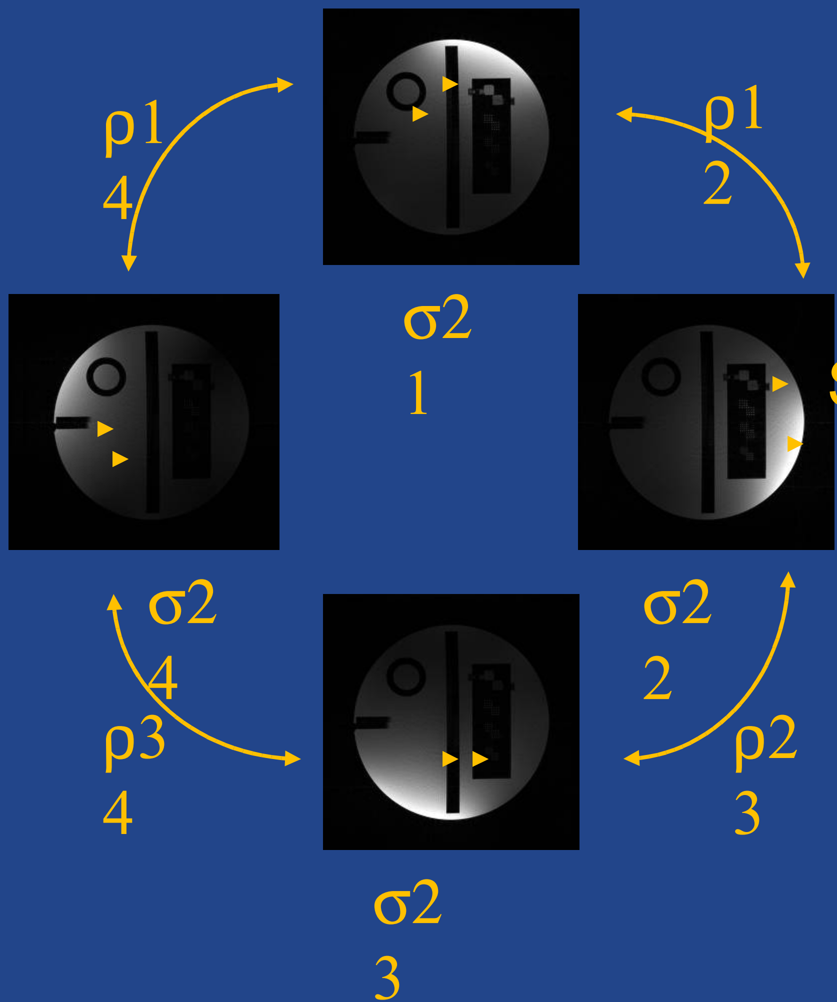
- Same variance of noise in each coil
- No correlation between coils
- No acceleration
- Reconstruction done with sum of squares

Real acquisitions.

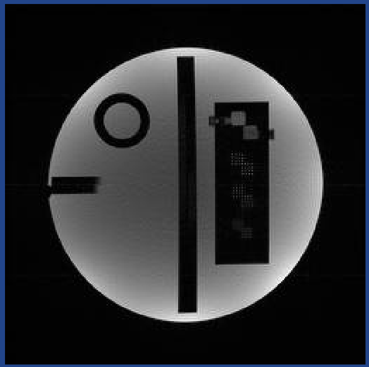
- Correlated, different variances
- Accelerated
- Reconstructed with different methods

A- Effect of Correlations

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1L} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{L1} & \sigma_{L2} & \cdots & \sigma_L^2 \end{pmatrix}$$



Sum of Squares →

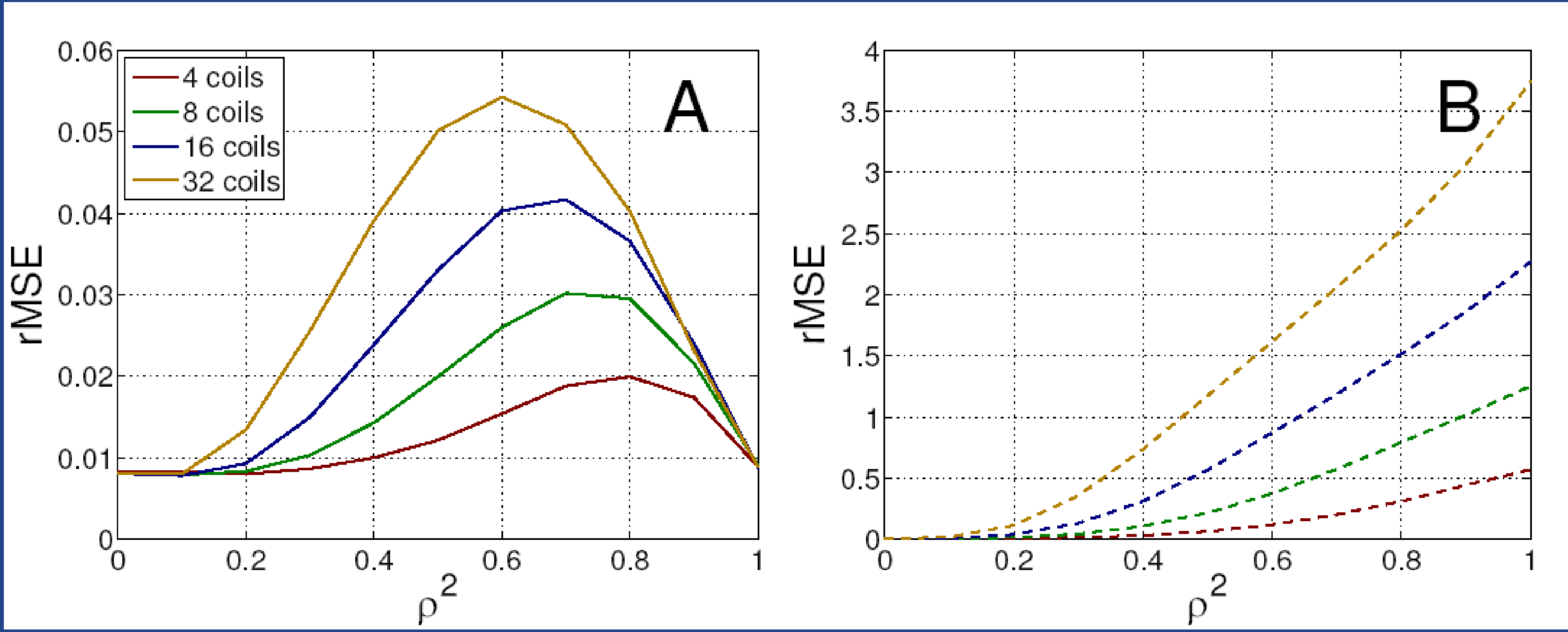


Nc-chi no longer valid!!!

$$L_{\text{eff}}(\mathbf{x}) = \frac{A_T^2(\mathbf{x})\operatorname{tr}(\boldsymbol{\Sigma}^2) + \left(\operatorname{tr}(\boldsymbol{\Sigma}^2)\right)^2}{\mathbf{A}^*(\mathbf{x})\boldsymbol{\Sigma}^2\mathbf{A}(\mathbf{x}) + \|\boldsymbol{\Sigma}^2\|_F^2};$$

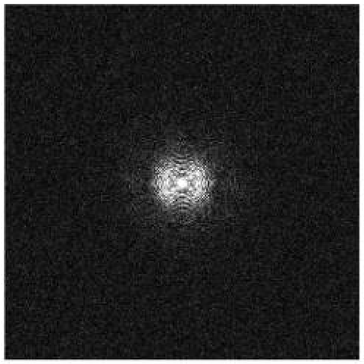
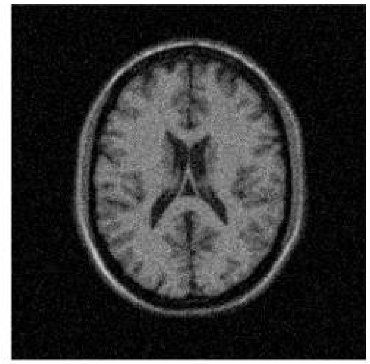
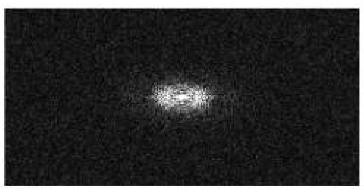
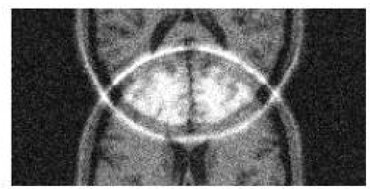
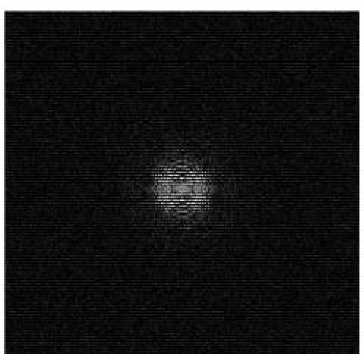
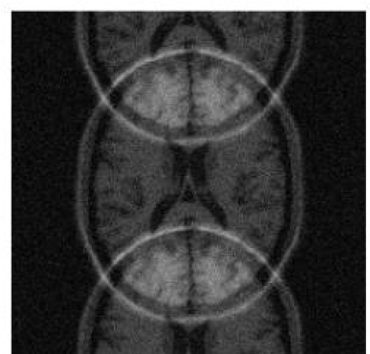
$$\sigma_{\text{eff}}^2(\mathbf{x}) = \frac{\operatorname{tr}(\boldsymbol{\Sigma}^2)}{L_{\text{eff}}(\mathbf{x})},$$

Nc-chi approximation if *effective* parameters are used



Relative errors in the PDF for central-chi approximation as a function of the correlation coefficient. A) Using effective parameters. B) Using the original parameters.

B- Sampling of the k-space:

k-space	Parameters	**x**-space	Relation
	Fully sampled, $\sigma^2_{K_l}$ **k**-size: $\lvert\Omega\rvert$		$\sigma_l^2 = \frac{1}{\lvert\Omega\rvert}\sigma^2_{K_l}$, **x**-size: $\lvert\Omega\rvert$
	Subsampled r, $\sigma^2_{K_l}$ **k**-size: $\lvert\Omega\rvert/r$		$\sigma_l^2 = \frac{r}{\lvert\Omega\rvert}\sigma^2_{K_l}$, **x**-size: $\lvert\Omega\rvert/r$
	Subsampled r, $\sigma^2_{K_l}$ **k**-size: $\lvert\Omega\rvert$ (zero padded)		$\sigma_l^2 = \frac{1}{\lvert\Omega\rvert\cdot r}\sigma^2_{K_l}$, **x**-size: $\lvert\Omega\rvert$

In real acquisitions:

- Different variances and correlations → effective values and approximated PDF.
- Subsampling → modification of $\sigma 2i$ in **x**-space
- Reconstruction method → output may be Rician or an approximation of nc-chi
- The variance of noise ($\sigma 2i$) becomes **x**-dependant → $\sigma 2i(x)$

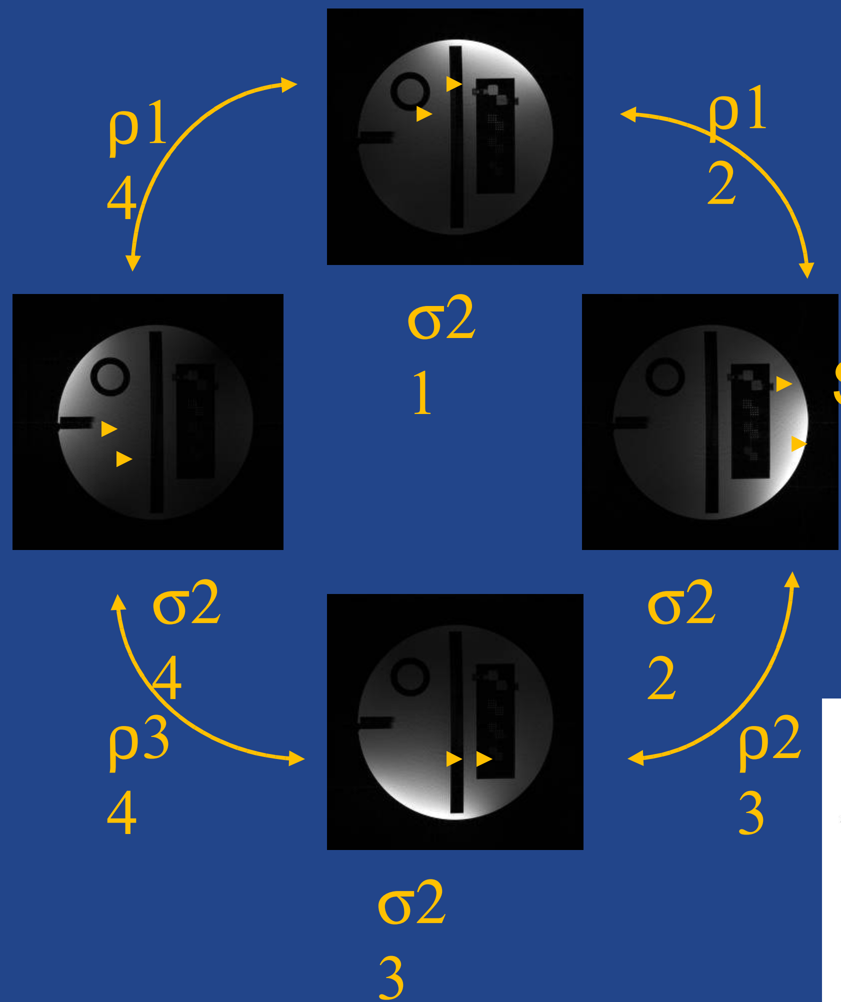
Composite Magnitude Signal				
Number of coils	**Acquisition**	**Statistical Model**	**Stationarity**	**Parameters**
1 coil	Single coil	Rician	Stationary	σ^2
Multiple coils (Uncorrelated)	No subsampling + SoS	nc-χ	Stationary	σ^2 L (Number of coils)
Multiple coils (correlated)	No subsampling + SoS	nc-χ (approx.)	Non-stationary	$\sigma^2_{\text{eff}}(\mathbf{x})$ $L_{\text{eff}}(\mathbf{x})$
Multiple coils	pMRI + SENSE	Correlated Rician	Non-stationary	$\sigma^2_{\mathcal{R}}(\mathbf{x})$ $\rho^2_{i,j}(\mathbf{x})$
Multiple coils	pMRI + GRAPPA+ SoS	nc-χ (approx.)	Non-stationary	$\sigma^2_{\text{eff}}(\mathbf{x})$ $L_{\text{eff}}(\mathbf{x})$

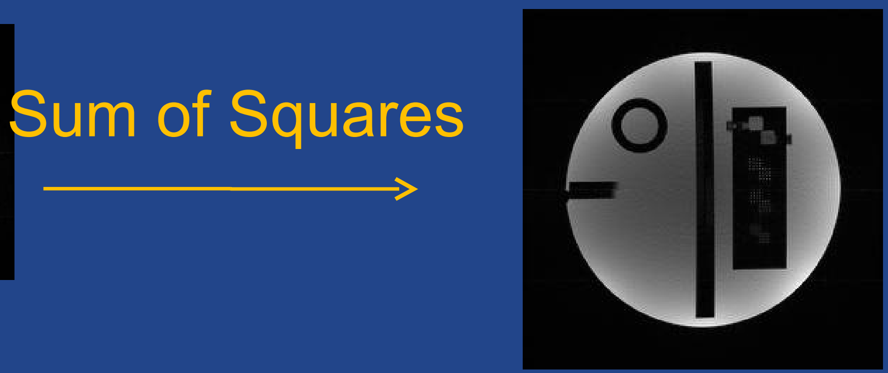
Survey of statistical models for MRI

4- Example: the non stationary nc-chi approximation

Nc-chi approximation

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1L} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{L1} & \sigma_{L2} & \cdots & \sigma_L^2 \end{pmatrix}$$



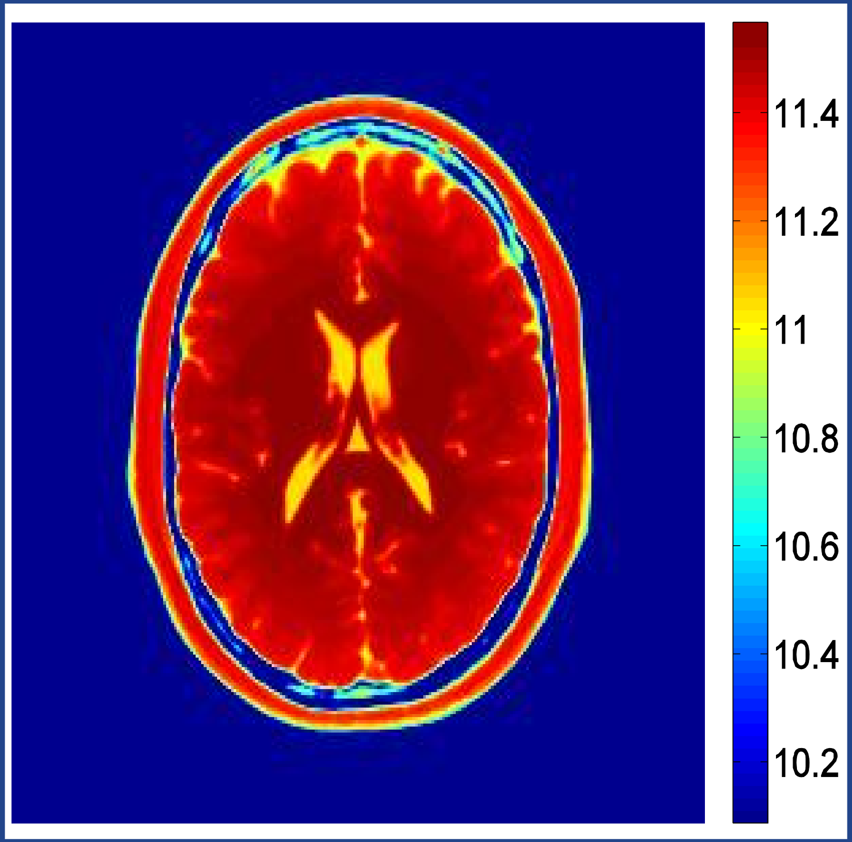


$$L_{\mathrm{eff}}(\mathbf{x}) = \frac{A_T^2(\mathbf{x})\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right) + \left(\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\right)^2}{\mathbf{A}^*(\mathbf{x})\boldsymbol{\Sigma}^2\mathbf{A}(\mathbf{x}) + \|\boldsymbol{\Sigma}^2\|_F^2};$$

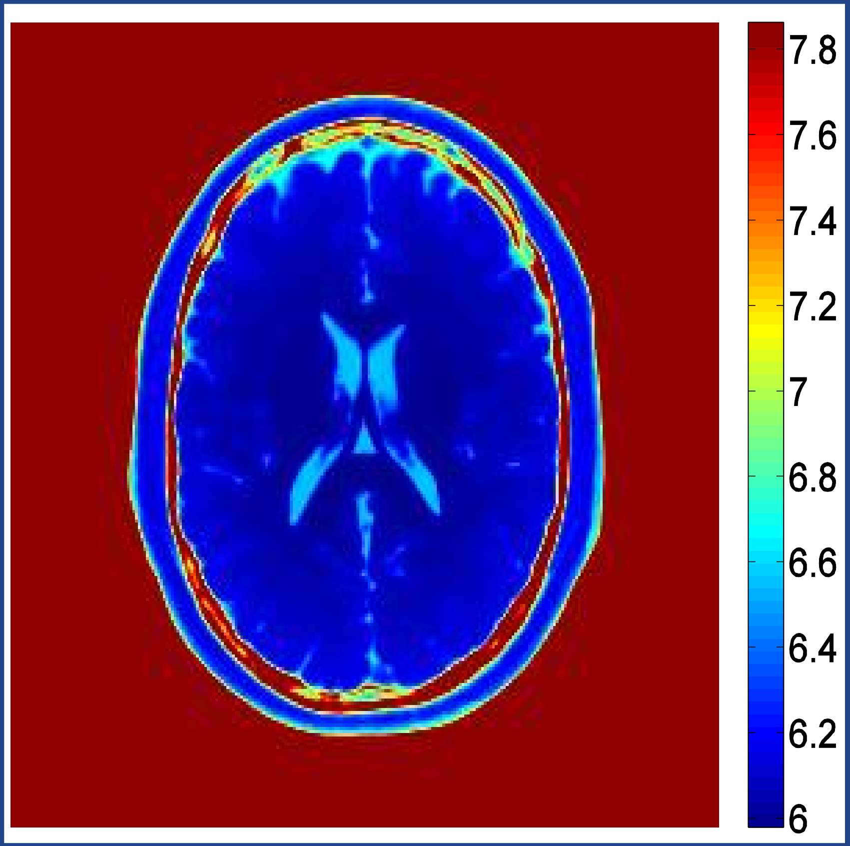
$$\sigma_{\mathrm{eff}}^2(\mathbf{x}) = \frac{\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)}{L_{\mathrm{eff}}(\mathbf{x})},$$

Nc-chi approximation if *effective* parameters are used

Effective number of coils as a function of the coefficient of correlation.
A) Absolute value. B) Relative Value.



$\sigma_{eff}(x)$



$L_{eff}(x)$

- Params. $\sigma_{eff}(x)$ and $L_{eff}(x)$ both depend on x.

- Good news:

$$L_{eff}(x) \cdot \sigma^2_{eff}(x) = tr(\boldsymbol{\Sigma}) = \sigma^2_1 + \ldots + \sigma^2_L = L \cdot \langle \sigma^2_i \rangle$$

- The product is a constant:

$$L_{eff}(x) \cdot \sigma^2_{eff}(x) = L \cdot \sigma^2_n$$

Implications:

- Equivalence between *effective* and *real* parameters
- Product: easy to estimate

$$L_{eff}(x)\cdot \sigma^2_{eff}(x) = L\cdot\sigma^2_n = \text{mode}\left\{ E\{M_L^2(x)\}_x \right\} / 2$$

- In some problems: only product needed:

$$I^2(x) = E\{M_L^2(x)\}_x - 2L\cdot\sigma^2_n$$

- NOTE: If effective values are used for $\sigma^2_{eff}(x)$, also for $L_{eff}(x)\cdot$

$$\sigma^2_{eff}(x)\cdot L \rightarrow$$ Wrong!!!

Implications:

- My point of view: $\sigma 2 \mathit{eff}(\mathrm{x})$ and $\mathrm{L}\mathit{eff}(\mathrm{x})$ should be seen as a single parameter:

$$\sigma 2 L = \mathrm{L}\mathit{eff}(\mathrm{x}) \cdot \sigma 2 \mathit{eff}(\mathrm{x})$$

and therefore

$$\sigma 2 \mathit{eff}(\mathrm{x}) = \sigma 2 L \,/\, \mathrm{L}\mathit{eff}(\mathrm{x})$$

- Some applications do need $\sigma 2 \mathit{eff}(\mathrm{x})$

$$K_L(\mathbf{x}) = 1 - \frac{4\sigma_n^2\left(\langle M_L^2(\mathbf{x})\rangle_{\mathbf{x}} - L\sigma_n^2\right)}{\langle M_L^4(\mathbf{x})\rangle_{\mathbf{x}} - \langle M_L^2(\mathbf{x})\rangle_{\mathbf{x}}^2}.$$

X-dependant noise:

- Param. $\sigma 2 \mathit{eff}(x)$ now depends on position.
- The dependence can be bounded.

SNR $\rightarrow$ 0

$$\sigma 2 B = \sigma 2 n (1 + \langle \rho 2 \rangle (L-1))$$

SNR $\rightarrow \infty$

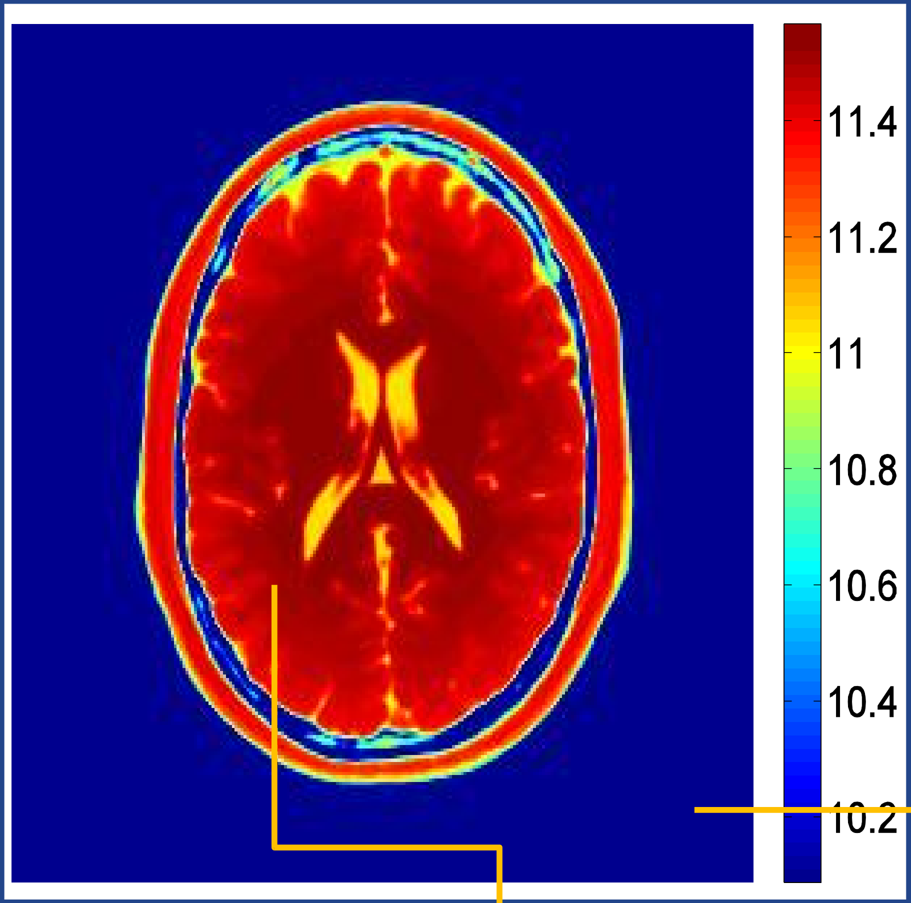
$$\sigma 2 S = \sigma 2 n (1 + \langle \rho \rangle (L-1))$$

Total:

$$\sigma 2 \mathit{eff}(x) = (1 - \phi(x))\, \sigma 2 S + \phi(x)\, \sigma 2 B$$

with

$$\phi(x) = (\mathrm{SNR}2(x) + 1) - 1$$



Where is noise estimated?

What to estimate: $\sigma 2 \mathit{eff}$(x), $\sigma 2 n$, $\sigma 2 B$, $\sigma 2 S$…

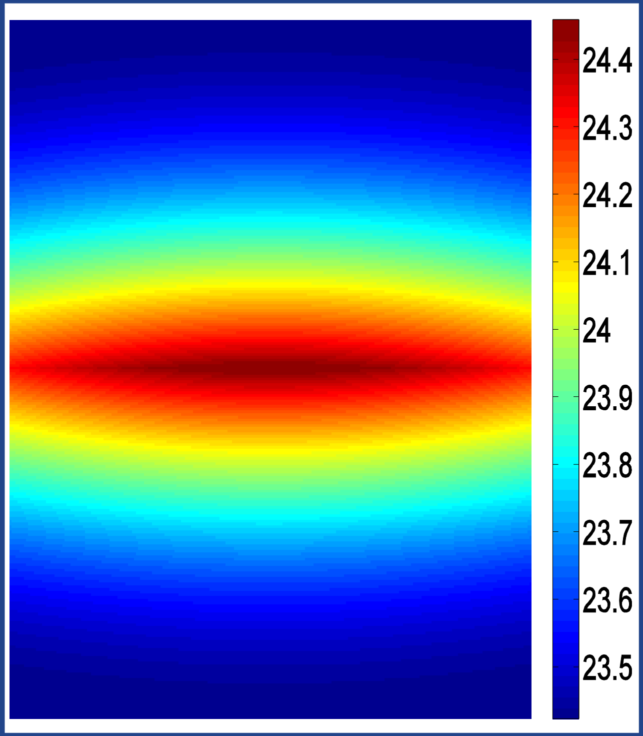
$\sigma 2 B$

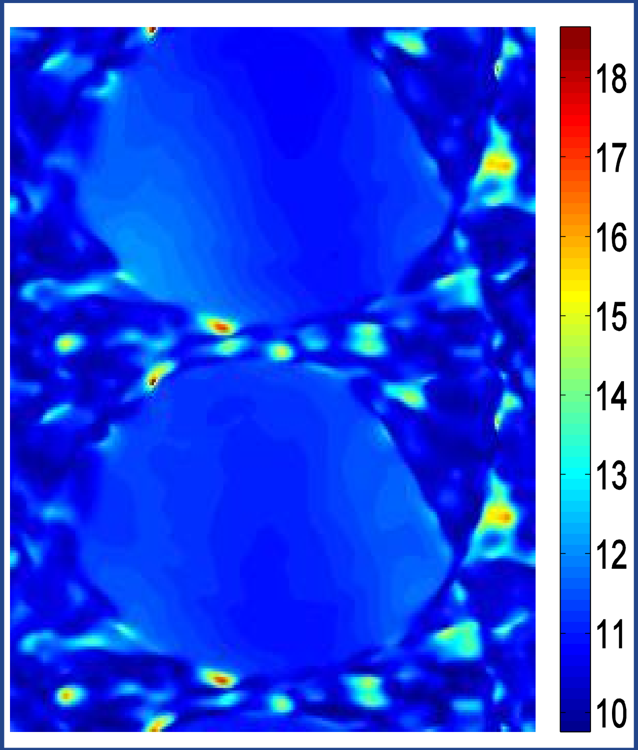
$\sigma 2 S$

Implications:

- Estimation over the background → underestimation of noise.
- Use of a single value of σn → error is most areas.
- Noise is higher in the high SNR.
- Main source of non-stationarity: correlation between coils.
- High correlation: lower effective coils and higher noise.
- Possible source of error: $\sigma 2 eff(x) \cdot L$

5- Other models

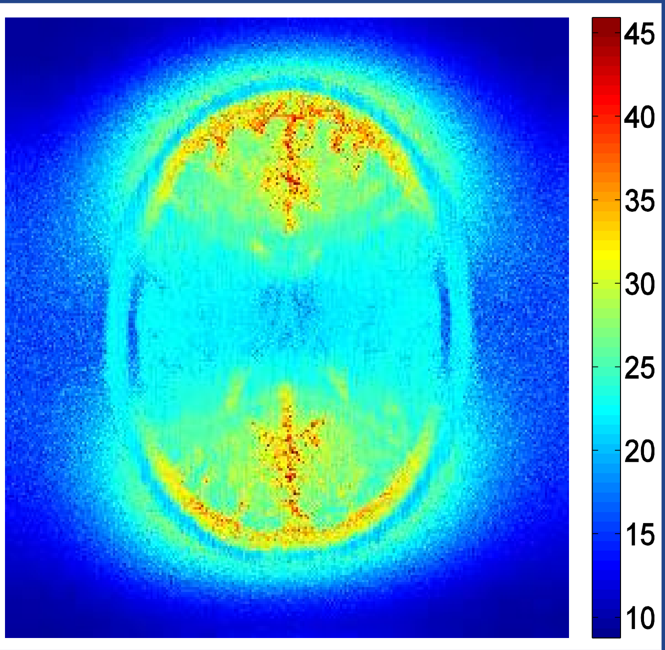


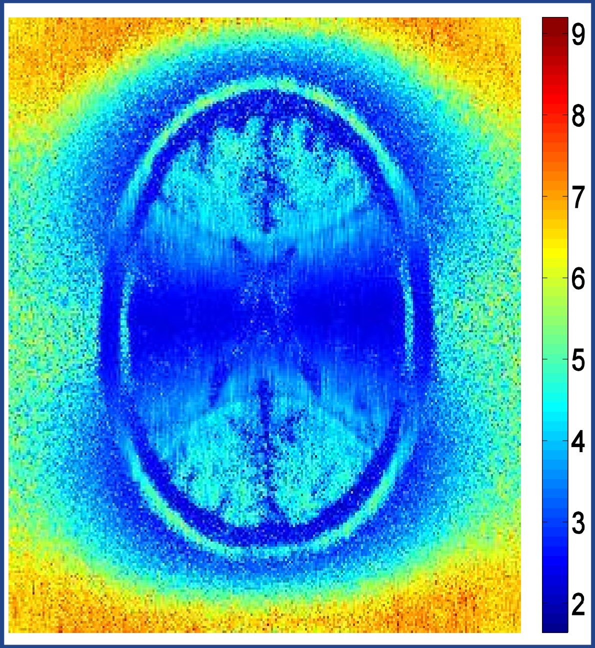


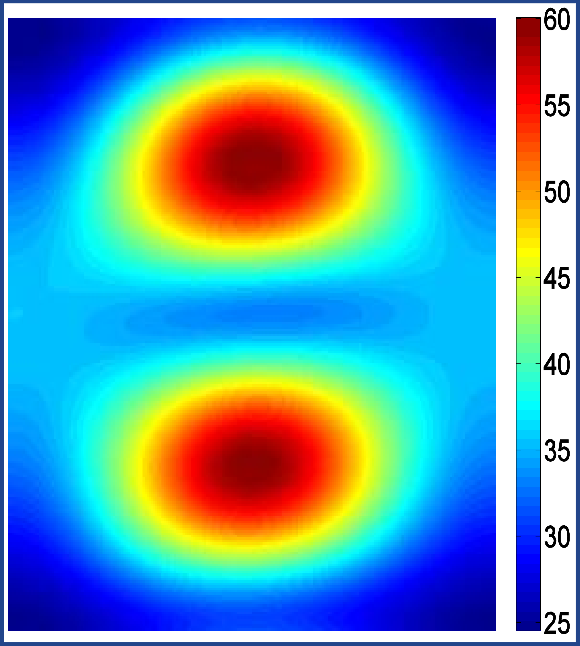
SENSE: (non stationary Rician)

$\sigma 2R$(x) x-dependant noise

$\sigma 2n = \sigma 2K$ (r / | Ω|)







$\sigma_{eff}(x)$ $L_{eff}(x)$ $\left(\sigma^2_{eff}(x)\, L_{eff}(x)\right)^{1/2}$

GRAPPA: nc-chi approx.

$$\sigma^2_n = \sigma^2_K \,/\, (r\,|\Omega|)$$

Product $\sigma^2_{eff}(x)\cdot L_{eff}(x)$ is not a constant

6- Conclusions

Conclusions:

- Be sure what you need in your application.
- Follow the whole reconstruction pipeline to be sure which is your “original” noise.
- Useful: simplified models to make process easier.
- Be sure how noise affects the different slides.

Questions?

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What to measure when measuring noise in MRI

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